SINGULAR SYMPLECTIC FLOPS AND RUAN COHOMOLOGY

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Abstract. In this paper, we study the symplectic geometry of singular conifolds of the finite group quotient

$$W_r = \{(x,y,z,t)|xy - z^{2r} + t^2 = 0\}/\mu_r(a,-a,1,0), r \geq 1,$$

which we call orbi-conifolds. The related orbifold symplectic conifold transition and orbifold symplectic flops are constructed. Let X and Y be two symplectic orbifolds connected by such a flop. We study orbifold Gromov-Witten invariants of exceptional classes on X and Y and show that they have isomorphic Ruan cohomologies. Hence, we verify a conjecture of Ruan.

1. Introduction

In [LR], the authors proved an elegant result that any two smooth minimal models in dimension three have the same quantum cohomology. Besides the key role of the relative invariants introduced in the paper, one of the main building blocks towards this result is the understanding of how the Gromov-Witten invariants change under flops. The description of a smooth flop is closely related to the conifold singularity

$$W_1 = \{(x,y,z,t)|xy - z^2 + t^2 = 0\}.$$

A crucial step in their proof is a symplectic description of a flop and hence symplectic techniques can be applied. However, it is well-known that the appropriate category for birational geometry is singular manifolds with terminal singularities. In complex dimension three, terminal singularities are deformations of orbifolds. In this paper and its sequel, we initiate a program to study the quantum cohomology under birational transformation of orbifolds.

In the singular category,

$$W_r = \{(x,y,z,t)|xy - z^{2r} + t^2 = 0\}/\mu_r(a,-a,1,0).$$

is a natural replacement for the smooth conifold. The orbifold symplectic flops coming from this model are defined in the first part of the

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paper (cf. §4). In the second part of the paper, we compute the 3-point function of (partial) orbifold Gromov-Witten invariants. This enables us to verify a conjecture by Ruan in the current set-up: for any two symplectic orbifolds X and Y connected via orbifold symplectic flops, their Ruan cohomology rings are isomorphic.

1.1. **Orbifold symplectic flops.** The singularity given by W_1 has been studied intensively. Let ω^o be the symplectic form on $W_1 \setminus \{0\}$ induced from that of $\mathbb{C}^4$. It has two small resolutions, denoted by W_1^s and W_1^{sf}, and a smoothing via deformation which is denoted by Q_1. The transformations

$$W_1^s \leftrightarrow Q_1, \quad W_1^{sf} \leftrightarrow Q_1$$

are called *conifold transitions*. And the transformation

$$W_1^s \leftrightarrow W_1^{sf}$$

is called a *flop*.

A symplectic conifold([STY]) (Z, ω) is a space with conifold singularities

$$P = \{p_1, \ldots\}$$

such that $(Z \setminus P, \omega)$ is a symplectic manifold and ω coincides with ω^o locally at $p_i \in P$. Now suppose that Z is compact and $|P| = \kappa < \infty$. Such Z admits a smoothing, denoted by X, and 2^κ resolutions

$$\mathcal{Y} = \{Y_1, \ldots, Y_{2^\kappa}\}.$$

In X each p_i is replaced by an exceptional sphere $L_i \cong S^3$, while for each Y_j, p_i is replaced by an extremal ray $\mathbb{P}^1$.

In [STY], they studied a necessary and sufficient condition for the existence of a symplectic structure on one of the Y in $\mathcal{Y}$ in terms of certain topological condition on X. They showed that *one of the 2^κ small resolutions admits a symplectic stucture if and only if on X we have the following homology relation*

$$[\sum_{i=1}^{\kappa} \lambda_i L_i] = 0 \in H_3(X, \mathbb{Z}) \;\; with \; \lambda_i \neq 0 \; for \; all \; i. \tag{1.1}$$

Here the L_i are exceptional spheres on X.

One can rephrase their theorem using cohomological language. Then, equation (1.1) reads as

$$[\sum_{i=1}^{\kappa} \lambda_i \Theta_i] = 0 \in H^3(X, \mathbb{Z}) \;\; with \; \lambda_i \neq 0 \; for \; all \; i. \tag{1.2}$$

Here Θ_i is the Thom form of the normal bundle of L_i.

The cohomological version will be generalized to the general model with finite group quotient. Our model is

$$W_r = \{(x, y, z, t) | xy - z^{2r} + t^2 = 0\}/\mu_r(a, -a, 1, 0), r \geq 1. \tag{1.3}$$

(see [K] and [Reid] for references). Such a local model is called *r-conifold or an orbi-conifold* in our paper. Such (terminal) singularities appear naturally in the Minimal Model Program. They are the simplest examples in the list of singularities in [K]. W_r without the finite quotient has been considered in [La]. It also has two resolutions $\tilde{W}_r^s$ and $\tilde{W}_r^{sf}$. We can take quotients

$$W_r^s = \tilde{W}_r^s/\mu_r, \;\; W_r^{sf} = \tilde{W}_r^{sf}/\mu_r.$$

Both of them are orbifolds. In this paper, we propose a smoothing Q_r as well. The transformations

$$W_r^s \leftrightarrow Q_r, \;\; W_r^{sf} \leftrightarrow Q_1$$

are called *(orbi)-conifold transitions.* And the transformation

$$W^s \leftrightarrow W^{sf}$$

is called a *(orbi)-flop.*

We are interested in symplectic geometry of the orbi-conifold (Z, ω_Z). It has a smoothing X and 2^κ small resolutions

$$\mathcal{Y} = \{Y_i, 1 \leq i \leq 2^\kappa.\}$$

A theorem generalizing that of Smith-Thomas-Yau is

Theorem 1.1. *One of the 2^κ small resolutions admits a symplectic stucture if and only if on X we have the following cohomology relation*

$$[\sum_{i=1}^{\kappa} \lambda_i \Theta_{r_i}] = 0 \in H^3(X, \mathbb{R}) \;\; with \; \lambda_i \neq 0 \; for \; all \; i. \tag{1.4}$$

As a corollary of this theorem, we show that if one of $Y_i \in \mathcal{Y}$ is symplectic then so is its flop $Y_i^f \in \mathcal{Y}$ (refer to §4.1 for the definition).

1.2. **The ring structures and Ruan's conjecture.** Let X be an orbifold. It is well known that $H^*(X)$ does not suffice for quantum cohomology. One should consider the so-called twisted sectors $X_{(g)}$ on X and study a bigger space

$$H^*_{CR} := H^*(X) \oplus \bigoplus_{(g)|g \neq 1} H^*(X_{(g)}).$$

Using the orbifold Gromov-Witten invariants [CR2], one can define the orbifold quantum ring $QH^*_{CR}(X)$. The analogue of classical cohomology is known as the Chen-Ruan orbifold cohomology ring.

Motivated by the work of Li-Ruan ([LR]) on the transformation of the quantum cohomology rings with respect to a smooth flop, we may ask how the orbifold quantum cohomology ring transforms (or even how the orbifold Gromov-Witten invariants change) via orbi-conifold transitions or orbifold flops. It can be formulated as the following conjecture

Conjecture 1.2. *Let Y be the orbifold symplectic flop of X, then*

$$QH^*_{CR}(X) \cong QH^*_{CR}(Y).$$

To completely answer the question, one needs a full package of technique, such as relative orbifold Gromov-Witten invariants and degeneration formulae. These techniques are out of reach at this moment and will be studied in future papers([CLZZ]).

On the other hand, it is easy to show that

$$H^*_{CR}(X) \cong H^*_{CR}(Y)$$

additively. In general, they will have different ring structures. In this paper, we study a new ring structure that it is in a sense between H^*_{CR} and QH^*_{CR}. It was first introduced by Ruan [**?**] in the smooth case and can be naturally extended to orbifolds. Let's review the construction. Let $\Gamma^s_i, \Gamma^{sf}_i, 1 \leq i \leq \kappa$ be extremal rays in X and Y respectively. On X, (and on Y), we use only moduli spaces of J-curves representing multiples of Γ_i's and define 3-point functions on $H^*_{CR}(X)$ by

$$\Psi^X_{qc}(\beta_1,\beta_2,\beta_3) = \Psi^X_{d=0}(\beta_1,\beta_2,\beta_3) + \sum_{i=1}^{\kappa}\sum_{d=1}^{\infty}\Psi^X_{(d[\Gamma^s],0,3)}(\beta_1,\beta_2,\beta_3). \tag{1.5}$$

Such functions also yield a product on $H^*_{CR}(X)$. This ring is called the Ruan cohomology ring [HZ] and denoted by $RH^*_{CR}(X)$. Ruan conjectures that *if X, Y are K-equivalent, $RH^*_{CR}(X)$ is isomorphic to $RH^*_{CR}(Y)$.*

Our second theorem is

Theorem 1.3. *Suppose that X and Y are connected by a sequence of symplectic flops constructed out of r-conifolds. Then $RH^*_{CR}(X)$ is isomorphic to $RH^*_{CR}(Y)$. Hence, Ruan's conjecture holds in this case.*

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2. Local Models

2.1. **Local r-orbi-conifolds.** Let

$$\mu_r = \langle \xi \rangle, \xi = e^{\frac{2\pi i}{r}}$$

be the cyclic group of r-th roots of 1. We denote its action on $\mathbb{C}^4$ by $\mu_r(a, b, c, d)$ if the action is given by

$$\xi \cdot (x, y, z, t) = (\xi^a x, \xi^b y, \xi^c z, \xi^d t).$$

Let $\tilde{W}_r \subset \mathbb{C}^4$ be the complex hypersurface given by

$$\tilde{W}_r = \{(x, y, z, t) | xy - z^{2r} + t^2 = 0\}, r \geq 1.$$

It has an isolated singularity at the origin. We call $\tilde{W}_r$ the *local r-conifold.* Set

$$\tilde{W}_r^\circ = \tilde{W}_r \setminus \{0\}.$$

It is clear that, for any integer a that is prime to r, the action $\mu_r(a, -a, 1, 0)$ preserves $\tilde{W}_r$. Set

$$W_r = \tilde{W}_r / \mu_r, \;\; W_r^\circ = \tilde{W}_r^\circ / \mu_r.$$

We call W_r the *local r-orbi-conifold.* Let $\tilde{\omega}_{r,w}^\circ$ be the symplectic structure on $\tilde{W}_r^\circ$ induced from $\mathbb{C}^4$. It yields a symplectic structure $\omega_{r,w}^\circ$ on W_r°.

2.2. **The small resolutions of W_r and flops.** By blow-ups, we have two small resolutions of $\tilde{W}_r$. They are

$$\begin{aligned} \tilde{W}_r^s &= \{((x, y, z, t), [p, q]) \in \mathbb{C}^4 \times \mathbb{P}^1 \\ &\qquad | xy - z^{2r} + t^2 = 0, \; \frac{p}{q} = \frac{x}{z^r - t} = \frac{z^r + t}{y}\} \\ \tilde{W}_r^{sf} &= \{((x, y, z, t), [p, q]) \in \mathbb{C}^4 \times \mathbb{P}^1 \\ &\qquad | xy - z^{2r} + t^2 = 0, \; \frac{p}{q} = \frac{x}{z^r + t} = \frac{z^r - t}{y}\}. \end{aligned}$$

Let

$$\tilde{\pi}_r^s : \tilde{W}_r^s \to W_r^s, \quad \tilde{\pi}_r^{sf} : \tilde{W}_r^{sf} \to W_r^{sf}$$

be the projections. The extremal rays $(\tilde{\pi}_r^s)^{-1}(0)$ and $(\tilde{\pi}_r^{sf})^{-1}(0)$ are denoted by $\tilde{\Gamma}_r^s$ and $\tilde{\Gamma}_r^{sf}$ respectively. Both of them are isomorphic to $\mathbb{P}^1$. The action of μ_r extends naturally to both resolutions by setting

$$\xi \cdot [p, q] = [\xi^a p, q]$$

for the first model and

$$\xi \cdot [p, q] = [\xi^{-a} p, q]$$

for the second one.

Set
$$W^s_r = \tilde{W}^s_r/\mu_r, \quad W^{sf}_r = \tilde{W}^{sf}_r/\mu_r \quad \Gamma^s_r = \tilde{\Gamma}^s_r/\mu_r \quad \Gamma^{sf}_r = \tilde{\Gamma}^{sf}_r/\mu_r.$$
We call W^s and W^{sf} *small resolutions* of W_r. We say that W^{sf} is the flop of W^s and vice versa. They are both orbifolds with singular points on Γ^s and Γ^{sf}.

Another important fact we use in this paper is

Proposition 2.1. *For $r \geq 2$, the normal bundle of $\tilde{\Gamma}^s_r$ $(\tilde{\Gamma}^{sf}_r)$ in $\tilde{W}^s_r$ $(\tilde{W}^{sf}_r)$ is $\mathcal{O} \oplus \mathcal{O}(-2)$.*

Proof. The proof is given in [La].

2.3. **Orbifold structures on W^s and W^{sf}.** Let us take W^s. The singular points are points 0 and ∞ on Γ^s. In term of $[p,q]$ coordinates, they are
$$0 = [0,1]; \quad \infty = [1,0].$$
We denote them by $\mathfrak{p}^s$ and $\mathfrak{q}^s$ respectively. Since $\tilde{W}^s \subset \mathbb{C}^5$ near $\mathfrak{p}^s$, the (tangent) of a uniformizing system of $\mathfrak{p}^s$ is given by
$$\{(p,x,y,z,t)|x = t = 0\}.$$
μ_r acts on this space by
$$\xi(p,y,z) = (\xi^a p, \xi^{-a} y, \xi z).$$
At $\mathfrak{p}^s$, for each given $\xi^k = \exp(2\pi ik/r), 1 \leq k \leq r$, there is a corresponding twisted sector([CR1]). As a set, it is same as $\mathfrak{p}^s$. We denote this twisted sector by $[\mathfrak{p}^s]_k$. For each twisted sector, a degree shifting number is assigned. We conclude that

Lemma 2.2. *For $\xi^k = \exp(2\pi ik/r), 1 \leq k \leq r$, the degree shifting*
$$\iota([\mathfrak{p}^s]_k) = 1 + \frac{k}{r}.$$

Proof. This follows directly from the definition of degree shifting. q.e.d.

Similar results hold for the singular point $\mathfrak{q}^s$. Hence we also have twisted sector $[\mathfrak{q}^s]_k$ and
$$\iota([\mathfrak{q}^s]_k) = 1 + \frac{k}{r}.$$

A similar structure applies to W^{sf}. There are two singular points, denoted by $\mathfrak{p}^{sf}, \mathfrak{q}^{sf}$. The corresponding twisted sectors are $[\mathfrak{p}^{sf}]_k, [\mathfrak{q}^{sf}]_k$. Then
$$\iota([\mathfrak{p}^{sf}]_k) = \iota([\mathfrak{q}^{sf}]_k) = 1 + \frac{k}{r}.$$

2.4. **The deformation of W_r.** For convenience, we change coordinates:

$$x = z_1 + \sqrt{-1}z_2, \quad y = z_1 - \sqrt{-1}z_2, \quad z = \sqrt[2r]{-1}z_3, \quad t = z_4.$$

Thus in terms of the new coordinates $\tilde{W}_r$ is given by a new equation

$$z_1^2 + z_2^2 + z_3^{2r} + z_4^2 = 0. \tag{2.1}$$

It is also convenient to use real coordinates

$$(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (z_1, z_2, z_3, z_4).$$

In terms of real coordinates, $\mu_r(a, -a, 1, 0)$ action is given by

$$e^{\frac{2\pi i}{r}} \cdot \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi a}{r} & 0 & -\sin\frac{2\pi a}{r} & 0 \\ 0 & \cos\frac{2\pi a}{r} & 0 & -\sin\frac{2\pi a}{r} \\ \sin\frac{2\pi a}{r} & 0 & \cos\frac{2\pi a}{r} & 0 \\ 0 & \sin\frac{2\pi a}{r} & 0 & \cos\frac{2\pi a}{r} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix},$$

and

$$e^{\frac{2\pi i}{r}} \cdot \begin{pmatrix} x_3 \\ y_3 \\ x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi}{r} & -\sin\frac{2\pi}{r} & 0 & 0 \\ \sin\frac{2\pi}{r} & \cos\frac{2\pi}{r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ x_4 \\ y_4 \end{pmatrix}.$$

The equation for $\tilde{W}_r$ is

$$\begin{cases} x_1^2 + x_2^2 + f^2(x_3, y_3) + x_4^2 = y_1^2 + y_2^2 + g^2(x_3, y_3) + y_4^2 \\ x_1y_1 + x_2y_2 + f(x_3, y_3)g(x_3, y_3) + x_4y_4 = 0. \end{cases}$$

Here f and g are defined by

$$f(x, y) + \sqrt{-1}g(x, y) = (x + \sqrt{-1}y)^r.$$

We propose

Definition 2.1. *The* deformation *of $\tilde{W}_r$ is the set $\tilde{Q}_r$ defined by*

$$\begin{cases} x_1^2 + x_2^2 + f^2(x_3, y_3) + x_4^2 = 1, \\ x_1y_1 + x_2y_2 + f(x_3, y_3)g(x_3, y_3) + x_4y_4 = 0. \end{cases}$$

The action $\mu_r(a, -a, 1, 0)$ preserves $\tilde{Q}_r$. Hence we set

$$Q_r = \tilde{Q}_r/\mu_r$$

and called it the deformation of W_r.

Lemma 2.3. *$\tilde{Q}_r$ is a 6-dimensional symplectic submanifold of* $\mathbb{R}^4 \times \mathbb{R}^4$.

Proof. Consider the map

$$\begin{aligned} F: \ & \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^2 \\ & (x,y) \to (F_1(x,y), F_2(x,y)). \end{aligned}$$

given by

$$\begin{aligned} F_1(x,y) &= x_1^2 + x_2^2 + f^2(x_3,y_3) + x_4^2 - 1, \\ F_2(x,y) &= x_1y_1 + x_2y_2 + f(x_3,y_3)g(x_3y_3) + x_4y_4. \end{aligned}$$

Then $F^{-1}(0) = \tilde{Q}_r$. The Jacobian of F is

$$\begin{pmatrix} 2x_1 & 2x_2 & 2f\frac{\partial f}{\partial x_3} & 2x_4 & 0 & 0 & 2f\frac{\partial f}{\partial y_3} & 0 \\ y_1 & y_2 & g\frac{\partial f}{\partial x_3} + f\frac{\partial g}{\partial x_3} & y_4 & x_1 & x_2 & g\frac{\partial f}{\partial y_3} + f\frac{\partial g}{\partial y_3} & x_4 \end{pmatrix}.$$

We claim that this is a rank 2 matrix: if one of x_1, x_2, x_4, say x_i, is nonzero, the above matrix has a rank 2 submatrix

$$\begin{pmatrix} 2x_i & 0 \\ y_i & x_i \end{pmatrix}.$$

Otherwise, say $(x_1, x_2, x_4) = (0,0,0)$; then by the definition of $\tilde{Q}_r$ we have $f(x_3,y_3) \neq 0$, and $g(x_3,y_3) = 0$. Then since $f + \sqrt{-1}g$ is a holomorphic function of $x_3 + \sqrt{-1}y_3$, we have

$$\begin{vmatrix} 2f\frac{\partial f}{\partial x_3} & 2f\frac{\partial f}{\partial y_3} \\ g\frac{\partial f}{\partial x_3} + f\frac{\partial g}{\partial x_3} & g\frac{\partial f}{\partial y_3} + f\frac{\partial g}{\partial y_3} \end{vmatrix} = (\frac{\partial f}{\partial x_3})^2 + (\frac{\partial f}{\partial y_3})^2 \neq 0.$$

Hence F has rank 2 everywhere on $\tilde{Q}_r$ and 0 is its regular value. This implies that $\tilde{Q}_r$ is a smooth 6-dimensional submanifold of $\mathbb{R}^4 \times \mathbb{R}^4$.

Next we prove that $\tilde{Q}_r$ has a canonical symplectic structure $\omega_{\tilde{Q}_r}$ induced from

$$(\mathbb{R}^4 \times \mathbb{R}^4, \omega_o = -\Sigma dx_i \wedge dy_i).$$

It is sufficient to prove that

$$\omega_o(\nabla F_1, \nabla F_2) \neq 0.$$

By direct computations,

$$\begin{aligned} \nabla F_1 &= (2x_1, 2x_2, 2f\frac{\partial f}{\partial x_3}, 2x_4, 0, 0, 2f\frac{\partial f}{\partial y_3}, y_3), \\ \nabla F_2 &= (y_1, y_2, f\frac{\partial g}{\partial x_3} + g\frac{\partial f}{\partial x_3}, y_4, x_1, x_2, f\frac{\partial g}{\partial y_3} + g\frac{\partial f}{\partial y_3}, x_4), \end{aligned}$$

Therefore

$$\begin{aligned} -\omega_o(\nabla F_1, \nabla F_2) &= \sum dx_i(\nabla F_1)dy_i(\nabla F_2) - dx_i(\nabla F_2)dy_i(\nabla F_1) \\ &= 2x_1^2 + 2x_2^2 + 2f((\frac{\partial f}{\partial x_3})^2 + (\frac{\partial g}{\partial x_3})^2) + 2x_4^2 \neq 0. \end{aligned}$$

Hence $\tilde{Q}_r$ is a symplectic submanifold with a canonical symplectic structure induced from $\mathbb{R}^4 \times \mathbb{R}^4$. q.e.d.

We denote the symplectic structure by $\tilde{\omega}^\circ_{r,q}$.

Put

$$\tilde{L}_r := \{(x, y) \in \tilde{Q}_r | y_1 = y_2 = g(x_3, y_3) = y_4 = 0\}.$$

and set

$$\tilde{Q}^\circ_r = \tilde{Q}_r \setminus \tilde{L}_r.$$

The μ_r-action preserves $\tilde{L}_r$; we set

$$L_r = \tilde{L}_r/\mu_r, \;\; Q^\circ_r = \tilde{Q}^\circ_r/\mu_r.$$

L_r is the exceptional set in Q_r with respect to the deformation in the following sense:

Lemma 2.4. *There is a natural diffeomorphism between W°_r and Q°_r.*

Proof. We denote by $[x, y] \in W^\circ_r$ the equivalence class of $(x, y) \in \tilde{W}_r$ with respect to the quotient by μ_r.

For any $\lambda > 0$ we let $\tilde{W}_{r,\lambda} \subset \tilde{W}_r$ be the set of (x, y) satisfying

$$x_1^2 + x_2^2 + f^2(x_3, y_3) + x_4^2 = y_1^2 + y_2^2 + g^2(x_3, y_3) + y_4^2 = \lambda$$

and

$$x_1y_1 + x_2y_2 + f(x_3, y_3)g(x_3, x_3) + x_4y_4 = 0.$$

It is not hard to see that

- $\tilde{W}_{r,\lambda}$ is preserved by the μ_r action; set
$$W_{r,\lambda} = \tilde{W}_{r,\lambda}/\mu_r;$$
- $\tilde{W}^\circ_r$ is foliated by $\tilde{W}_{r,\lambda}, \lambda \in \mathbb{R}^+$.

On the other hand, $\tilde{Q}^\circ_r$ has a similar foliation: for $\lambda > 0$, let $\tilde{Q}_{r,\lambda} \subset \tilde{Q}_r$ be the set of (x, y) satisfying

$$\begin{aligned} &x_1^2 + x_2^2 + f^2(x_3, y_3) + x_4^2 = 1, \\ &y_1^2 + y_2^2 + g^2(x_3, y_3) + y_4^2 = \lambda^2, \\ &x_1y_1 + x_2y_2 + f(x_3, y_3)g(x_3, x_3) + x_4y_4 = 0. \end{aligned}$$

Then

- $\tilde{Q}_{r,\lambda}$ is preserved by the μ_r action; set
$$Q_{r,\lambda} = \tilde{Q}_{r,\lambda}/\mu_r;$$
- $\tilde{Q}^\circ_r$ is foliated by $\tilde{Q}_{r,\lambda}, \lambda \in \mathbb{R}^+$.

We next introduce the identification between $W_{r,\lambda}$ and $Q_{r,\lambda}$. Let $u_\lambda(x_3, y_3)$ and $v_\lambda(x_3, y_3)$ be functions that solve
$$(u + iv)^r = \lambda^{-1} f(x_3, y_3) + \sqrt{-1}\lambda g(x_3, y_3).$$
Such a pair $u + iv$ exists up to a factor ξ^k. Then
$$[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4] \longleftrightarrow$$
$$[\lambda^{-1}x_1, \lambda^{-1}x_2, u(x_3, y_3), \lambda^{-1}x_4, \lambda y_1, \lambda y_2, v(x_3, y_3), \lambda y_4]$$
induces an identification between $W_{r,\lambda}$ and $Q_{r,\lambda}$, and therefore between W°_r and Q°_r. q.e.d.

We denote the identification map constructed in the proof by
$$\Phi_r : W^\circ_r \to Q^\circ_r.$$
In particular, we note that the restriction of Φ_r to $W_{r,1}$ is the identity.

2.5. **The comparison between local r-orbi-conifolds and local conifolds.** When $r = 1$, the local model is the well-known conifold. Since $\mu_r = \mu_1 = \{1\}$ is trivial, there is no orbifold structure. It is well known that

- W^s_1 and W^{sf}_1 are
$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1,$$
where Γ^s and Γ^{sf} are the zero section $\mathbb{P}^1$; They are flops of each other;
- Q_1 is diffeomorphic to the cotangent bundle of S^3. The induced symplectic structure from $\mathbb{R}^4 \times \mathbb{R}^4$ coincides with the canonical symplectic structure on T^*S^3.
- the map
$$\Phi_1 : (W_1, \omega^\circ_{1,w}) \to (Q_1, \omega^\circ_{1,q})$$
is a symplectomorphism.

There are natural (projection) maps
$$\pi_{r,w} : \tilde{W}_r \to W_1, \quad \pi_{r,q} : \tilde{Q}_r \to Q_1$$
given by
$$x_i \to x_i, \quad y_i \to y_i, i \neq 3,$$
and
$$(x_3, y_3) \to (f(x_3, y_3), g(x_3, y_3)).$$

Similarly, there are maps

$$\pi^s_{r,w} : \tilde{W}^s_r \longrightarrow W^s_1, \ \ \pi^{sf}_{r,w} : \tilde{W}^{sf}_r \longrightarrow W^{sf}_1.$$

We note that all these projection maps are almost r-coverings. They are coverings except on $x_3 = y_3 = 0$, where the maps are only r-branched coverings. Note that

$$\tilde{L}_r = \pi^{-1}_{r,q} L_1.$$

It is the union of r copies of S^3 intersecting at

$$\left\{ \begin{array}{l} x_1^2 + x_2^2 + x_4^2 = 1 \\ x_1 y_1 + x_2 y_2 + x_4 y_4 = 0 \end{array} \right\} \cap \{x_3 = y_3 = 0\}.$$

3. Cohomologies

3.1. **Definitions.** Let $(\Omega^*(\tilde{W}^\circ_r), d)$ be the de Rham complex of $\tilde{W}^\circ_r$. μ_r has a natural representation on this complex. let

$$\Omega^*_{\mu_r}(\tilde{W}^\circ_r) \subset \Omega^*(\tilde{W}^\circ_r)$$

be the subcomplex of μ_r-invariant forms. We have

$$H^*(W^\circ_r) = H^*(\Omega^*_{\mu_r}(\tilde{W}^\circ_r), d).$$

Similar definitions apply to $W^s_r, W^{sf}_r, Q^\circ_r, Q_r, W_{r,1} = Q_{r,1}$ etc.

Then

Lemma 3.1. *$H^*(W^\circ_r) = H^*(W_{r,1})$.*

Proof. We note that there is a μ_r-isomorphism

$$\tilde{W}^\circ_r \cong \tilde{W}_{r,1} \times \mathbb{R}^+.$$

In fact, it is induced by a natural identification

$$\begin{array}{rcl} \tilde{W}_{r,\lambda} & \leftrightarrow & \tilde{W}_{r,1} \times \{\lambda\}; \\ x_i & \leftrightarrow & \lambda^{-\frac{1}{2}} x_i, i \neq 3; \ x_3 \leftrightarrow \lambda^{-\frac{1}{2r}} x_3, \\ y_i & \leftrightarrow & \lambda^{-\frac{1}{2}} y_i, i \neq 3; \ y_3 \leftrightarrow \lambda^{-\frac{1}{2r}} y_3. \end{array}$$

Hence $\tilde{W}^\circ_r$ is μ_r-homotopy equivalent to $\tilde{W}_{r,1}$. Hence the claim follows. q.e.d.

The result also follows from

$$W^\circ_r \cong W_{r,1} \times \mathbb{R}^+$$

directly. Similarly, we have

$$Q^\circ_r \cong Q_{r,1} \times \mathbb{R}^+.$$

Hence

$$H^*(Q^\circ_r) = H^*(Q_{r,1}).$$

Note that $Q_{r,1} = W_{r,1}$. We have

$$H^*(W_r^\circ) = H^*(W_{r,1}) = H^*(Q_{r,1}) = H^*(Q_r^\circ).$$

3.2. **Computation of cohomologies.** We first study $H^*(W_{r,1})$.

Recall that we have a map

$$\pi_{r,w} : \tilde{W}_{r,1} \to W_{1,1}$$

given by

$$\pi_{r,w}(x,y) = (x_1, x_2, f(x_3,y_3), x_4, y_1, y_2, g(x_3,y_3), y_4).$$

We now introduce a μ_r action on $W_{1,1}$. For convenience, we use coordinates (u,v) for the $\mathbb{R}^4 \times \mathbb{R}^4$ in which $W_{1,1}$ is embedded. Then

$$e^{\frac{2\pi i}{r}} \cdot \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi a}{r} & 0 & -\sin\frac{2\pi a}{r} & 0 \\ 0 & \cos\frac{2\pi a}{r} & 0 & -\sin\frac{2\pi a}{r} \\ \sin\frac{2\pi a}{r} & 0 & \cos\frac{2\pi a}{r} & 0 \\ 0 & \sin\frac{2\pi a}{r} & 0 & \cos\frac{2\pi a}{r} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix},$$

and acts trivially on u_3, v_3, u_4 and v_4. Then it is clear that $\pi_{r,w}$ is μ_r-equivariant. It induces a morphism between complexes

$$\pi_{r,w}^* : (\Omega^*_{\mu_r}(W_{1,1}), d) \to (\Omega^*_{\mu_r}(\tilde{W}_{r,1}), d). \tag{3.1}$$

Here Ω_G always represents the subspace that is G-invariant if Ω is a G-representation.

Proposition 3.2. *$\pi_{r,w}^*$ in (3.1) is an isomorphism between the* cohomologies *of the two complexes.*

Proof. The idea of the proof is to consider a larger connected Lie group action on spaces: Let $S^1 = \{e^{2\pi i\theta}\}$. Suppose its action on (x,y) is given by

$$e^{2\pi i\theta} \cdot \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix},$$

and the trivial action on x_3, y_3, x_4 and y_4. The same action is defined on (u,v). Again, $\pi_{r,w}$ is S^1-equivariant.

Since S^1 is a connected Lie group and its actions commutes with μ_r-actions on both spaces, the subcomplex

$$((\Omega^*_{\mu_r}(\tilde{W}_{r,1}))_{S^1}, d) \subset (\Omega^*_{\mu_r}(\tilde{W}_{r,1}), d)$$

of S^1-invariant forms yields same cohomology as the original one, i.e,

$$H^*((\Omega^*_{\mu_r}(\tilde{W}_{r,1}))_{S^1}, d) = H^*(\Omega^*_{\mu_r}(\tilde{W}_{r,1}), d)$$

Similarly,

$$H^*((\Omega^*_{\mu_r}(W_{1,1}))_{S^1}, d) = H^*(\Omega^*_{\mu_r}(W_{1,1}), d)$$

It is then sufficient to show that

$$\pi^*_{r,w} : H^*((\Omega^*_{\mu_r}(W_{1,1}))_{S^1}, d) \to H^*((\Omega^*_{\mu_r}(\tilde{W}_{r,1}))_{S^1}, d) \tag{3.2}$$

is an isomorphism. By the definition of the actions, we note that

$$(\Omega^*_{\mu_r}(W_{1,1}))_{S^1} = \Omega^*_{S^1}(W_{1,1}). \tag{3.3}$$

We now show (3.2). Recall that $\pi_{r,w}$ is an r-branched covering ramified over

$$R_1 = \left\{ \begin{array}{l} u_1^2 + u_2^2 + u_4^2 = v_1^2 + v_2^2 + v_4^2 = 1 \\ u_1 v_1 + u_2 v_2 + u_4 v_4 = 0 \end{array} \right\} \cap \{u_3 = v_3 = 0\}$$

Set $\tilde{R}_r = \pi^{-1}_{r,w}(R_1)$ and

$$\tilde{U}_r = \tilde{W}_{r,1} \setminus \tilde{R}_r, \quad U_1 = W_{1,1} \setminus R_1.$$

Then $\pi_{r,w} : \tilde{R}_r \to R_1$ is 1-1 and $\pi_{r,w} : \tilde{U}_r \to U_1$ is an$r$-covering.

Let V_1 be an S^1-invariant tubular neighborhood of R_1 in $W_{1,1}$. By the implicit function theorem, we know that

$$V_1 \cong R_1 \times D_1,$$

where D_1 is the unit disk in the complex plane $\mathbb{C} = \{u_3 + \sqrt{-1} v_3\}$. Let $\tilde{V}_r = \pi^{-1}_{r,w}(V_1)$. Then

$$\tilde{V}_r \cong \tilde{R}_r \times D_1,$$

where D_1 is the unit disk in the complex plane $\mathbb{C} = \{x_3 + iy_3\}$. In terms of these identifications, $\pi_{r,w}$ can be rewritten as

$$\begin{aligned} &\pi_{r,w} : \tilde{R}_r \times D_1 \to R_1 \times D_1 \\ &\pi_{r,w}(\gamma, z_3) = (\gamma, z_3^r), \end{aligned}$$

where $\gamma \in \tilde{R}_r = R_1$, $z_3 = x_3 + iy_3$.

Consider the short exact sequences

$$0 \to (\Omega^*_{\mu_r}(W_{1,1}))_{S^1} \to (\Omega^*_{\mu_r}(U_1))_{S^1} \oplus (\Omega^*_{\mu_r}(V_1))_{S^1} \to (\Omega^*_{\mu_r}(U_1 \cap V_1))_{S^1} \to 0$$

and

$$0 \to (\Omega^*_{\mu_r}(\tilde{W}_{r,1}))_{S^1} \to (\Omega^*_{\mu_r}(\tilde{U}_r))_{S^1} \oplus (\Omega^*_{\mu_r}(\tilde{V}_r))_{S^1} \to (\Omega^*_{\mu_r}(\tilde{U}_r \cap \tilde{V}_r))_{S^1} \to 0.$$

$\pi^*_{r,w}$ is a morphism between two complexes. We assert that

$$\pi^*_{r,w} : H^*((\Omega^*_{\mu_r}(U_1))_{S^1}, d) \xrightarrow{\cong} H^*((\Omega^*_{\mu_r}(\tilde{U}_r))_{S^1}, d), \tag{3.4}$$

$$\pi^*_{r,w} : H^*((\Omega^*_{\mu_r}(V_1))_{S^1}, d) \xrightarrow{\cong} H^*((\Omega^*_{\mu_r}(\tilde{V}_r))_{S^1}, d), \tag{3.5}$$

$$\pi^*_{r,w} : H^*((\Omega^*_{\mu_r}(U_1 \cap V_1))_{S^1}, d) \xrightarrow{\cong} H^*((\Omega^*_{\mu_r}(\tilde{U}_r \cap \tilde{V}_1))_{S^1}, d). \tag{3.6}$$

Once these are proved, by the five-lemma, we know that

$$\pi_{r,w}^* : H^*((\Omega^*_{\mu_r}(W_{1,1}))_{S^1}, d) \xrightarrow{\cong} H^*((\Omega^*_{\mu_r}(\tilde{W}_{r,1}))_{S^1}, d)$$

which is (3.2).

We now explain (3.4), (3.5) and (3.6).

The proof of (3.4). We observe that

$$\pi_{r,w}^* : (\Omega^*_{\mu_r}(U_1))_{S^1} \xrightarrow{\cong} (\Omega^*_{\mu_r}(\tilde{U}_r))_{S^1}.$$

Hence it induces an isomorphism on cohomology level.

The proof of (3.5). Since $\tilde{V}_r$ is $\mu_r \times S^1$-homotopy equivalent to $\tilde{R}_r$, we have

$$H^*((\Omega^*_{\mu_r}(\tilde{V}_r))_{S^1}, d) \cong H^*((\Omega^*_{\mu_r}(\tilde{R}_r))_{S^1}, d).$$

Similarly,

$$H^*((\Omega^*_{\mu_r}(V_1))_{S^1}, d) \cong H^*((\Omega^*_{\mu_r}(R_1))_{S^1}, d).$$

Because

$$H^*((\Omega^*_{\mu_r}(\tilde{R}_r))_{S^1}, d) = H^*((\Omega^*_{\mu_r}(R_1))_{S^1}, d),$$

we have (3.5).

The proof of (3.6). The proof is the same as that of (3.4).

This completes the proof of the theorem. q.e.d.

So far, we have shown that

$$H^*(W_{r,1}) = H^*(\Omega^*_{\mu_r}(\tilde{W}_{r,1}), d) \cong H^*(\Omega^*_{\mu_r}(W_{1,1}), d) = H^*((\Omega^*_{\mu_r}(W_{1,1}))_{S^1}, d).$$

Furthermore, by (3.3) we have

$$H^*((\Omega^*_{\mu_r}(W_{1,1}))_{S^1}, d) = H^*(\Omega^*_{S^1}(W_{1,1}), d) = H^*(W_{1,1}).$$

Since $W_{1,1} \cong S^3 \times S^2$ we have

Corollary 3.3. *$H^*(W_{r,1}) \cong H^*(S^3 \times S^2)$.*

Let H_1 be a generator of $H^2(S^3 \times S^2)$ such that

$$\int_{S^2} H_1 = 1.$$

Here S^2 is any fiber $\{x\} \times S^2$ in $S^3 \times S^2$. Set

$$\tilde{H}_r = \pi_{r,w}^* H_1$$

and let H_r be its induced form on $W_{r,1}$. This is a generator of $H^2(W_{r,1})$. Without loss of generality, we also assume that it is a generator of $H^2(W_r^\circ)$.

Let $\omega_{r,w}$ and $\omega_{r,q}$ be symplectic forms on W_r° and Q_r° respectively. Suppose that

$$[\omega_{r,w}|_{W_{r,1}}] = [\omega_{r,q}|_{Q_{r,1}}].$$

Here $[\omega]$ denotes the cohomology class of ω. Then there exists a symplectomorphism

$$\Phi_r' : (W_r^\circ, \omega_{r,w}) \to (Q_r^\circ, \omega_{r,q}).$$

In fact, by the assumption, we have

$$[\omega_{r,w}] = [\Phi_r^* \omega_{r,q}].$$

Then, by the standard Moser argument, there exists a diffeomorphism

$$f : W_r^\circ \to W_r^\circ$$

such that $f^*\omega_{r,w} = \Phi_r^*\omega_{r,q}$. Now we can set $\Phi_r' = \Phi_r \circ f^{-1}$. In particular, by applying it to $\omega_{r,w}^\circ$ and $\omega_{r,q}^\circ$ we have

Corollary 3.4. *There exists a symplectomorphism*

$$\Phi_r' : (W_r^\circ, \omega_{r,w}^\circ) \to (Q_r^\circ, \omega_{r,q}^\circ).$$

Proof. We observe that both symplectic forms are exact. Hence they represent the same cohomology class, namely 0. q.e.d.

Next we consider $H^*(W_r^s)$. The argument is same as above: we also have a map

$$\pi_{r,w} : \tilde{W}_r^s \to W_1^s.$$

This map will induce an isomorphism

Proposition 3.5. $H^*(W_r^s) = H^*(W_1^s)$.

Proof. Since the proof is parallel to that of proposition 3.2, we only sketch the proof.

We use complex coordinates $(x, y, z, t, [p, q])$ for $\tilde{W}_r^s$ and $(u, v, w, s, [m, n])$ for W_1^s. Then $\pi_{r,w}$ is induced by the map

$$u = x,\ v = y,\ w = z^r,\ s = t,\ \frac{m}{n} = \frac{p}{q}.$$

We can introduce a μ_r-action on W_1^s by

$$\xi(u, v, w, s, [m, n]) = (\xi^a u, \xi^{-a} v, w, s, [\xi^a m, n]), \xi = e^{\frac{2\pi i}{r}}.$$

Then $\pi_{r,w}$ is μ_r-equivariant.

Moreover, both spaces admit an S^1-action such that $\pi_{r,w}$is S^1-equivariant: for $\xi \in S^1$:

$$\begin{aligned} \xi(x, y, z, t, [p, q]) &= (\xi^a x, \xi^{-a} y, z, t, [\xi^a p, q]) \\ \xi(u, v, w, s, [m, n]) &= (\xi^a u, \xi^{-a} v, w, s, [\xi^a m, n]). \end{aligned}$$

$\pi_{r,w}$ is an r-branched covering ramified over

$$W_1^s \cap \{w = 0\}.$$

Then the rest of the proof is simply a copy of the argument in Proposition 3.2. q.e.d.

Since

$$W_1^s \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1),$$

$H^2(W_1^s) = H^2(\mathbb{P}^1)$ is 1-dimensional. So is $H^2(W_r^s)$. Let H_r^s be the generator of $H^2(W_r^s)$ such that

$$\int_{\Gamma_r^s} H_r^s = 1.$$

Since the normal bundle of $\tilde{\Gamma}_r^s$ is $\mathcal{O} \oplus \mathcal{O}(-2)$, it admits a symplectic form ω'. We normalize it by

$$\int_{\Gamma_r^s} \omega' = 1.$$

It induces a symplectic structure, denoted by ω_r^s on the neighborhood U of Γ_r^s. It is easy to see that this symplectic structure is tamed by the complex structure on U. Hence we conclude that

Corollary 3.6. *There is a symplectic form on W_r^s that represents the class H_r^s and is tamed by its complex structure. This form is denoted by ω_r^s.*

4. Orbifold symplectic flops

4.1. **The global orbi-conifolds.** Following [STY] we give the definition of orbi-conifolds.

Definition 4.1. *A* real 6-dimensional orbi-conifold *is a topological space Z covered by an atlas of charts $\{(U_i, \phi_i)\}$ of the following two types: either (U_i, ϕ_i) is an orbifold chart or*

$$\phi_j : U_j \to W_{r_j}$$

is a homeomorphism onto W_{r_j} defined in §2.1. In the latter case, we call the point $\phi_j^{-1}(0)$ a singularity of Z.

Moreover, the transition maps $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ must be smooth in the orbifold sense away from singularities and if $p \in U_i \cap U_j$ is a singularity then we have $r_i = r_j$ (denote it by r), and there must be an open subset $N \subset \mathbb{C}^4$ containing 0 such that the lifting of ϕ_{ij},

$$\tilde{\phi}_{ij} : \tilde{W}_r \cap N \longrightarrow \tilde{W}_r \cap N$$

in the uniformizing system is the restriction of an analytic isomorphism $\tilde{\phi} : \mathbb{C}^4 \to \mathbb{C}^4$ *which is smooth away from the origin,* C^1 *at the origin with* $d\tilde{\phi}_0 \in Sp(8, \mathbb{R})$*, and set-wise fixes* $\tilde{W}_r$*.*

We call such charts *smooth admissible coordinates.* Note that in the case $r = 1$ the singularity is the ordinary double point discussed in [STY].

¿From now on, we label the set of singularities

$$P = \{p_1, p_2, \ldots\},$$

and for each point p_i its local model is given by a standard model W_{r_i}.

Definition 4.2. *A* symplectic structure *on an orbi-conifold* Z *is a smooth orbifold symplectic form* ω_Z *on the orbifold* $Z \setminus P$ *which, around each singularity* p_i*, coincides with* ω°_{w,r_i}*.* (Z, ω_Z) *is called a* symplectic orbi-conifold.

¿From now on, we assume that Z is compact and $|P| = \kappa$. One can perform a smoothing for each singularity of Z as in §2.4 - replace a neighborhood of each singularity p_i by a neighborhood of L_{r_i} in Q_{r_i} - to get an orbifold. We denote this orbifold by X.

For each singularity p_i of Z we can perform two small resolutions, i.e., we replace the neighborhood of the singularity by $W^s_{r_i}$ or $W^{sf}_{r_i}$ as in §2.2. There are 2^κ choices of small resolutions, and so we get 2^κ orbifolds $Y_1, \cdots, Y_{2^\kappa}$.

Definition 4.3. *Two small resolutions* Y *and* Y' *are said to be* flops of each other *if at each* p_i*, one is obtained by replacing* $W^s_{r_i}$ *and the other by* $W^{sf}_{r_i}$*. We write* $Y' = Y^f$ *and vice versa.*

4.2. **Symplectic structures on Y_i's and flops.** Not every small resolution $Y_\alpha, 1 \le \alpha \le 2^\kappa$ admits a symplectic structure. Our first main theorem of the paper gives a necessary and sufficient condition for Y to have a symplectic structure in terms of the topology of X.

Let $L_{r_i} \subset X$. For simplicity, we assume its neighborhood to be Q_{r_i}. Recall that there is a projection map

$$\pi_{r_i,q} : \tilde{Q}_{r_i} \to Q_1.$$

Let Θ_1 be the Thom form of the normal bundle of $L_1 = S^3$ in Q_1. We assume it is supported in a small neighborhood of L_1. Set

$$\tilde{\Theta}_{r_i} = \pi^*_{r_i,q} \Theta_1.$$

We can choose Θ_1 properly such that $\tilde{\Theta}_{r_i}$ is μ_{r_i} invariant. Hence it induces a local form Θ_{r_i} on Q_{r_i} and hence on X.

Then we restate Theorem 1.1: *One of the* 2^κ *small resolutions admits a symplectic stucture if and only if on* X *we have the following cohomology relation*

$$[\sum_{i=1}^{\kappa}\lambda_i\Theta_{r_i}] = 0 \in H^3(X,\mathbb{R}) \ \ with\ \lambda_i \neq 0\ for\ all\ i. \tag{4.1}$$

As a corollary,

Corollary 4.1. *Suppose we have a pair of resolution* Y *and* Y^f *that are flops of each other. Then* Y *admits a symplectic structure if and only if* Y^f *does.*

Y^f is then called the *symplectic flop* of Y.

5. Proof of theorem 1.1

5.1. **Necessity.** We first prove that (4.1) is necessary.

Suppose that we have a Y that admits a symplectic structure ω. For simplicity, we assume that at each singular point $p_i \in Z$, it is replaced by $W^s_{r_i}$ to get Y. The extremal ray is Γ^s_i. Set

$$\lambda_i = \int_{\Gamma^s_{r_i}} \omega = \frac{1}{r_i}\int_{\tilde{\Gamma}^s_{r_i}} \tilde{\omega}.$$

Now we consider the pair of spaces $(X, X\backslash \cup L_{r_i})$. The exact sequence of the (orbifold) de Rham complex of the pair is

$$0 \to \Omega^{*-1}(X \setminus \cup L_{r_i}) \xrightarrow{\gamma} \Omega^*(X, X \setminus \cup L_{r_i}) \xrightarrow{\delta} \Omega^*(X) \to 0.$$

given by

$$\gamma(f) = (0, f), \quad \delta(\alpha, f) = \alpha.$$

It induces a long exact sequence on (orbifold) cohomology

$$\cdots \to H^2(X \setminus \cup L_{r_i}) \to H^3(X, X \setminus \cup L_{r_i}) \to H^3(X) \to \cdots$$

And applying this to ω on $Z \setminus P \cong X \setminus \cup_i L_{r_i}$, we have

$$\omega \mapsto (0, \omega) \mapsto 0.$$

This says that

$$[\delta \circ \gamma(\omega)] = 0.$$

We compute the left hand side of the equation. First, by applying the excision principle we get

$$H^3(X, X \setminus \cup_i L_{r_i}) \cong \bigoplus_i H^3(Q_{r_i}, Q^\circ_{r_i}).$$

This reduces the computation to the local case.

Let $\omega_{r_i,w}$ be the restriction of ω in the neighborhood, simply denoted by $W^s_{r_i}$, of $\Gamma^s_{r_i}$. It induces a form $\omega_{r_i,q}$ on $Q^\circ_{r_i}$. Suppose that

$$\omega_{r_i,q} = c_i H_{r_i},$$

where H_{r_i} is the generator on $Q_{r_i,1}$, hence on $Q^\circ_{r_i}$. Let β be a cut-off function such that

$$\beta(t) = \begin{cases} 1, & if \quad t > 0.5; \\ 0, & if \quad t < 0.25. \end{cases}$$

By direct computation, we have

$$\delta \circ \gamma(H_{r_i}) = d(\beta(\lambda)H_{r_i}) = \Theta_{r_i}.$$

Therefore, we conclude that

$$\sum_{i=1}^{\kappa} c_i \Theta_{r_i} = 0$$

In order to show (4.1), it remains to prove that

Proposition 5.1. *$c_i = -\lambda_i$.*

Proof. The computation is done on $\tilde{W}^s_{r_i}$.

Take an S^2 in $Q_{1,1}$ as

$$B_1 = \{(1, 0, 0, 0, 0, v_2, v_3, v_4) \in \tilde{Q}_{r_i} | v_2^2 + v_3^2 + v_4^2 = 1\}$$

Let $\tilde{B}_r = \pi^{-1}_{r,q}(B_1)$. It is

$$\tilde{B}_{r_i} = \{(1, 0, x_3, 0, 0, y_2, y_3, y_4) \in \tilde{Q}_{r_i} | y_2^2 + g^2(x_3, y_3) + v_4^2 = 1, f(x_3, y_3) = 0\}$$

Then

$$\int_{\tilde{B}_{r_i}} \tilde{H}_{r_i} = r_i \int_{B_1} H_1 = r_i.$$

Hence

$$\int_{\tilde{B}_{r_i}} \omega_{r_i,q} = c_i r_i.$$

Next we explain that

$$\int_{\tilde{B}_{r_i}} \omega_{r_i,q} = -\lambda_i r_i. \tag{5.1}$$

Then the claim follows from these two identities.

Proof of (5.1)*:* We treat B_1 and $\tilde{B}_{r_i}$ as subsets of W^s_1 and $\tilde{W}^s_{r_i}$. By Proposition 3.2, we assume $\omega_{r_i,w}$ is homologous to $\pi^*_{r_i,w}\omega$ for some $\omega \in H^2(W^s_1)$. Then

$$\int_{\tilde{B}_{r_i}} \omega_{r_i,q} = r_i \int_{B_1} \omega.$$

On the other hand, B_1 is homotopic to $-\Gamma_1^s$: via complex coordinates W_1 is given by
$$uv - (w-s)(w+s) = 0.$$
The equation of the small resolution W_1^s in the chart $\{q \neq 0\}$ is
$$\zeta v - (w-s) = 0, ,$$
where $\zeta = \frac{m}{n} = \frac{u}{w+s}$ is the coordinate of the exceptional curve Γ_1^s. Recall that on B_1 the complex coordinates are
$$x = 1 + y_2, \ \ y = 1 - y_2, \ \ z = \sqrt{-1}y_3, \ \ t = y_4.$$
We have a projection map
$$B_1 \longrightarrow \Gamma_1^s$$
given by
$$\eta = \frac{x}{z+t} = \frac{1+\sqrt{1-y_3^2-y_4^2}}{\sqrt{-1}y_3+y_4}.$$
Here we take $y_3,\ y_4$ as coordinates on B_1. It is easy to see that this is a one to one map and the point with $\sqrt{-1}y_3 + y_4 = 0$ corresponds to the point "∞" of $-\Gamma_1^s$. The sign is due to the orientation.

Let
$$(\zeta, y, z, t) = (\frac{1+\sqrt{1-y_3^2-y_4^2}}{\sqrt{-1}y_3+y_4}, 1-y_2, iy_3, y_4)$$
be any point in B_1; then
$$(\zeta_0, 0, 0, 0) = (\frac{1+\sqrt{1-y_3^2-y_4^2}}{\sqrt{-1}y_3+y_4}, 0, 0, 0)$$
is in Γ_1^s. We construct a subset Λ_1 of W_1^s
$$\rho(y_3, y_4, s) = \{(\frac{1+\sqrt{1-y_3^2-y_4^2}}{\sqrt{-1}y_3+y_4}, s(1-y_2), s\sqrt{-1}y_3, sy_4)\}$$
where $0 \leq s \leq 1$ and $y_3,\ y_4$ are the coordinates of N_1. This is a smooth 3-dimensional submanifold with boundary
$$\{\rho(y_3, y_4, 0) = -\Gamma_1^s\} \cup \{\rho(y_3, y_4, 1) = B_1\}.$$

It gives us a homotopy between $-\Gamma_1^s$ and B_1. Then
$$\int_{\tilde{B}_{r_i}} \omega_{r_i,w} = r_i \int_{B_1} \omega = -r_i \int_{\Gamma_1^s} \omega = -\int_{\tilde{\Gamma}_{r_i}^s} \omega_{r_i,w} = -r_i\lambda_i.$$
This shows (5.1).

We have completed the proof of the proposition. q.e.d.

This completes the proof of necessity.

Remark 5.2. *If the local resolution is $W^{sf}_{r_i}$,*

$$[\delta \circ \gamma(\omega_{r_i,w})] = \lambda_i \Theta_{r_i}.$$

5.2. **Sufficiency.** Suppose that (4.1) holds for X: i.e, there exists λ_i such that

$$\sum_i \lambda_i \Theta_{r_i} = 0.$$

For the moment we assume that all $\lambda_i < 0$. Let Y be a small resolution of Z obtained by replacing the neighborhood of p_i by $W^s_{r_i}$. We assert that Y admits a symplectic structure.

¿From the exact sequence of the pair of spaces $(X, X \setminus \cup_i L_{r_i})$

$$H^2(X \setminus \cup_i L_{r_i}) \xrightarrow{\gamma} H^3(X, X \setminus L_{r_i}) \to H^3(X)$$

we conclude that there exists a 2-form $\sigma^* \in H^2(X \setminus \cup_i L_{r_i})$ such that

$$\gamma(\sigma^*) = \sum \lambda_i \Theta_{r_i}.$$

since

$$X \setminus \cup_i L_{r_i} \cong Y \setminus \cup_i \Gamma^s_{r_i},$$

$\sigma^* \in H^2(Y \setminus \cup_i \Gamma^s_{r_i})$. On the other hand, we consider the exact sequence of the pair of spaces $(Y, Y \setminus \cup_i \Gamma^s_{r_i})$

$$H^2(Y) \to H^2(Y \setminus \cup_i \Gamma^s_{r_i}) \to H^3(Y, Y \setminus \cup \Gamma^s_{r_i}) \cong \bigoplus_i H^3(W^s_{r_i}, W^{\circ}_{r_i}).$$

It is known that locally $\tilde{W}^s_{r_i}$ is diffeomorphic to its normal bundle $\mathcal{O} \bigoplus \mathcal{O}(-2)$ of $\tilde{\Gamma}_{r_i}$, thus

$$H^3(Y, Y \setminus \cup \Gamma^s_{r_i}) = 0.$$

It follows that there exist a 2-form $\sigma \in H^2(Y)$ which extends σ^*.

Let U_i be a small neighborhood of $\Gamma^s_{r_i}$ in Y and $\tilde{U}_i \subset \tilde{W}^s_{r_i}$ be its pre-image in the uniformizing system. Set

$$\sigma_i = \sigma|_{U_i}.$$

By the proof of necessity, we know that

$$[\sigma_i] = [-\lambda_i \omega^s_{r_i}].$$

Then we can deform σ_i in its cohomology class near $\tilde{\Gamma}^s_{r_i}$ such that

$$\sigma_i = -\lambda_i \omega^s_{r_i}.$$

Hence we get a new form σ on Y that gives symplectic forms near Γ^s_i. On the other hand, we have a form ω_Z on Z that is symplectic away

from P. This form extends to Y, still denoted by ω_Z, but is degenerate at the $\Gamma^s_{r_i}$. For sufficiently large N we have

$$\Omega = \sigma + N\omega_Z.$$

This is a symplectic structure on Y: Ω is non-degenerate away from a small neighborhood of the $\Gamma^s_{r_i}$ for large N; both σ and ω_Z are tamed by the complex structure in the U_i, i.e,

$$\sigma(\cdot, J\cdot) > 0, \ \ \omega_Z(\cdot, J\cdot) \geq 0,$$

therefore

$$\Omega(\cdot, J\cdot) > 0,$$

which says that Ω is also a symplectic structure near the $\Gamma^s_{r_i}$. Hence (Y, Ω) is symplectic.

We now remark that the assumption on the sign of λ_i is inessential: suppose that $\lambda_1 > 0$; then we alter Y by replacing the neighborhood of p_1 by $W^{sf}_{r_1}$. Then the construction of the symplectic structure on this Y is the same.

5.3. **Proof of corollary 4.1.** This follows from remark 5.2. If Y and Y^f are a pair of flops, then one of them satisfies some equation

$$\sum_i \lambda_i \Theta_{r_i} = 0$$

and the other one satisfies

$$-\sum_i \lambda_i \Theta_{r_i} = 0.$$

Therefore, the symplectic structures exist on them simultaneously.

6. Orbifold Gromov-Witten invariants of W^s_r and W^{sf}_r

We first introduce the cohomology group for an orbifold in the stringy sense. Then we compute the orbifold Gromov-Witten invariants.

¿From now on, $r \geq 2$ is fixed. So we drop r from W^s_r and W^{sf}_r.

6.1. **Chen-Ruan orbifold cohomology of W^s and W^{sf}.** The stringy orbifold cohomology of W^s is

$$H^*_{CR}(W^s) = H^*(W^s) \oplus \bigoplus_k \mathbb{C}[\mathfrak{p}^s]_k \oplus \bigoplus_k \mathbb{C}[\mathfrak{q}^s]_k.$$

We abuse the notation here such that $[\mathfrak{p}^s]_k$ represents the 0-cohomology of the sector $[\mathfrak{p}^s]_k$. On the other hand, the grading should be treated carefully: the degree of an element in $H^*(W^s)$ remains the same, however the degree of $[\mathfrak{p}^s]_k$ is $0 + \iota([\mathfrak{p}^s]_k)$ and the same treatment applies to $[\mathfrak{q}^s]_k$. We call these new classes *twisted classes*.

A similar definition applies to W^{sf}.

$$H^*_{CR}(W^{sf}) = H^*(W^{sf}) \oplus \bigoplus_k \mathbb{C}[\mathfrak{p}^{sf}]_k \oplus \bigoplus_k \mathbb{C}[\mathfrak{q}^{sf}]_k.$$

6.2. **Moduli spaces $\overline{\mathcal{M}}_{0,l,k}(W^s, d[\Gamma^s], \mathbf{x}), k \geq 1$.** Here

$$\mathbf{x} = (T_1, \ldots, T_k)$$

consists of k twisted sectors in W^s.

By the definition in [CR2], the moduli space $\overline{\mathcal{M}}_{0,l,k}(W^s, d[\Gamma^s], \mathbf{x})$ consists of orbifold stable holomorphic maps from genus 0 curves, on which there are l smooth marked points and k orbifold points $y_1, \ldots, y_k$, to W^s such that

- y_i are sent to Y_i;
- the isotropy group at y_i is $\mathbb{Z}_{|\xi^a|}$ if $y_i = [p]_a$ (or $[q]_a$), where $|\xi^a|$ is the order of ξ^a;
- the image of the map represents the homology class $d[\Gamma^s]$.

By a genus 0 curve we mean S^2, or a bubble tree consisting of several S^2's. The stability is the same as in the smooth case.

Remark 6.1. *There is an extra feature for orbifold stable holomorphic maps. That is, the nodal points on a bubble tree may also be orbifold singular points on its component: for example, say y is a nodal point that is the intersection of two spheres S^2_+ and S^2_-; then y can be a singular points, denoted by y_+ and y_- respectively, on both spheres. Moreover if y_+ is mapped to $[p]_a$, y_- has to be mapped to $[p]_{r-a}$.*

When we write $\mathcal{M}_{0,l,k}(W^s, d[\Gamma^s], \mathbf{x})$, we mean the map whose domain is S^2. Usually, we call $\overline{\mathcal{M}}$ the compactified space of $\mathcal{M}$ and $\mathcal{M}$ the top stratum of $\overline{\mathcal{M}}$.

Lemma 6.2. *For $k \geq 1$, the virtual dimension*

$$\dim \overline{\mathcal{M}}_{0,0,k}(W^s, d[\Gamma^s], \mathbf{x}) < 0.$$

Proof. We recall that the virtual dimension is given by

$$2c_1(d[\Gamma^s]) + 2(n-3) + k - \sum_{i=1}^{k} \iota(Y_i) = k - \sum_{i=1}^{k} \iota(Y_i) < k - k = 0.$$

Here we use Lemma 2.2. q.e.d.

Lemma 6.3. $\mathcal{M}_{0,0,1}(W^s, d[\Gamma^s], \mathbf{x}) = \emptyset$.

Proof. This also follows from the dimension formula: the virtual dimension of this moduli space is a *rational* number. q.e.d.

6.3. **Moduli spaces $\overline{\mathcal{M}}_{0,0,0}(W^s, d[\Gamma^s])$.** Convention of notations: If $k = 0$, it is dropped and the moduli space is denoted by $\overline{\mathcal{M}}_{0,l}(W^s, d[\Gamma^s])$; if $k = l = 0$, then the moduli space is denoted by $\overline{\mathcal{M}}_0(W^s, d[\Gamma^s])$.

We have shown that $\overline{\mathcal{M}}_{0,0,k}(W^s, d[\Gamma^s], \mathbf{x})$ for $k \geq 1$ has some nice properties, following from the dimension formula. Now we focus on $k = 0$. Although its top stratum $\mathcal{M}_0(W^s, d[\Gamma^s])$ consists of only "smooth" maps, there may be orbifold maps in lower strata. Here, by the smoothness of a map we mean that the domain of the map is without orbifold singularities. The next lemma rules out this possibility.

Lemma 6.4. $\overline{\mathcal{M}}_0(W^s, d[\Gamma^s])$ *only consists of smooth maps.*

Proof. If not, suppose we have a map $f \in \overline{\mathcal{M}}_0(W^s, d[\Gamma^s])$ that consists of orbifold type nodal points in the domain. By looking at the bubble tree, we start searching from the leaves to look for the first component, say S^2_i, that containing a singular nodal point. This component *must* contain only *one* singular point. So $f|_{S^2_i}$ is an element in some moduli space $\mathcal{M}_{0,0,1}(W^s, d[\Gamma^s], \mathbf{x})$. But it is claimed in Lemma 6.3 that such an element does not exist. This proves the lemma. q.e.d.

Notice that $W^s = \tilde{W}^s/\mu_r$ and $\Gamma^s = \tilde{\Gamma}^s/\mu_r$. We may like to compare the moduli space $\overline{\mathcal{M}}_0(W^s, d[\Gamma^s])$ with $\overline{\mathcal{M}}_0(\tilde{W}^s, d[\tilde{\Gamma}^s])$. Note that μ_r acts naturally on the latter space. We claim that

Proposition 6.5. $\overline{\mathcal{M}}_0(W^s, d[\Gamma^s]) = \emptyset$ *if* $r \nmid d$. *Otherwise, there is a natural isomorphism*

$$\overline{\mathcal{M}}_0(W^s, mr[\Gamma^s]) = \overline{\mathcal{M}}_0(\tilde{W}^s, m[\tilde{\Gamma}^s])/\mu_r.$$

if $d = mr$.

Proof. Since

$$\overline{\mathcal{M}}_0(W^s, d[\Gamma^s]) = \overline{\mathcal{M}}_0(\Gamma^s, d[\Gamma^s])$$

and

$$\overline{\mathcal{M}}_0(\tilde{W}^s, d[\tilde{\Gamma}^s]) = \overline{\mathcal{M}}_0(\tilde{\Gamma}^s, d[\tilde{\Gamma}^s]),$$

it is sufficient to show that $\overline{\mathcal{M}}_0(W^s, d[\Gamma^s]) = \emptyset$ if $r \nmid d$ and

$$\overline{\mathcal{M}}_0(\Gamma^s, mr[\Gamma^s]) = \overline{\mathcal{M}}_0(\tilde{\Gamma}^s, m[\tilde{\Gamma}^s])/\mu_r.$$

We need the following lemma. Let $\pi : \tilde{\Gamma}^s \to \Gamma^s$ be the projection given by the quotient of μ_r. We claim that

Lemma 6.6. *for any smooth map*

$$f : S^2 \to \Gamma^s$$

there is a lifting $\tilde{f} : S^2 \to \tilde{\Gamma}^s$ *such that* $\tilde{\Pi}(\tilde{f}) = f$.

Now suppose the lemma is proved. Then we have that

$$\overline{\mathcal{M}}_0(W^s, d[\Gamma^s]) = \emptyset$$

for $r \nmid d$.

To prove the second statement, we define a map:

$$\tilde{\Pi} : \overline{\mathcal{M}}_0(\tilde{\Gamma}^s, m[\tilde{\Gamma}^s]) \to \overline{\mathcal{M}}_0(\Gamma^s, mr[\Gamma^s])$$

given by $\tilde{\Pi}(\tilde{f}) = \pi \circ \tilde{f}$. It is clear that this induces an injective map

$$\Pi : \overline{\mathcal{M}}_0(\tilde{\Gamma}^s, m[\tilde{\Gamma}^s])/\mu_r \to \overline{\mathcal{M}}_0(\Gamma^s, mr[\Gamma^s]).$$

On the other hand, since a stable smooth map on a bubble tree consists of smooth maps on each component of the tree that match at nodal points, therefore, by Lemma 6.6 the map can be components wise lifted. This shows that Π is surjective. q.e.d.

Proof of Lemma 6.6: S^2 and Γ^s are $\mathbb{P}^1$. We identify them as $\mathbb{C} \cup \{\infty\}$ as usual. On Γ^s, we assume $\mathfrak{p}^s$ and $\mathfrak{q}^s$ are 0 and ∞ respectively.

Suppose that

$$\Lambda_0 = f^{-1}(\mathfrak{p}^s) = \{x_1, \ldots, x_m\}, \quad , \Lambda_\infty = f^{-1}(\mathfrak{q}^s) = \{y_1, \ldots, y_n\}.$$

Let z be the coordinate of the first sphere; we write

$$f(z) = [p(z), q(z)].$$

Now since f is assumed to be smooth at the x_i, the map can be lifted with respect to the uniformizing system of $\mathfrak{p}^s$: namely, suppose that

$$\pi^s_{\mathfrak{p}} : D_\epsilon(0) \subset \mathbb{C} \to D_{\epsilon^r}(\mathfrak{p}^s)\mathbb{C}; \quad \pi^s_{\mathfrak{p}}(w) = w^r$$

gives the uniformizing system of the neighborhood of $\mathfrak{p}^s$ for some ϵ; f, restricted to a small neighborhood U_{x_i}, can be lifted to

$$\tilde{f} : U_{x_i} \to D_\epsilon$$

such that $f = \pi^s_{\mathfrak{p}} \circ \tilde{f}$. Without loss of generality, we assume that $f(U_{x_i}) = D_\epsilon(0)$. Therefore we have a lifting

$$\tilde{f} : \bigcup_i U_{x_i} \cup \bigcup_j U_{y_j} \to D_\epsilon(0) \cup D_\epsilon(\infty)$$

for f. Now we look at the rest of the map

$$f : S^2 - \bigcup_i U_{x_i} \cup \bigcup_j U_{y_j} \to \Gamma^s - D_{\epsilon^r}(\mathfrak{p}^s) \cup D_{\epsilon^r}(\mathfrak{q}^s).$$

We ask if this map can be lifted to the covering space

$$\tilde{\Gamma}^s - D_{\epsilon^r}(0) \cup D_{\epsilon^r}(\infty) \to \Gamma^s - D_{\epsilon^r}(\mathfrak{p}^s) \cup D_{\epsilon^r}(\mathfrak{q}^s).$$

The answer is affirmative by the elementary lifting theory for the covering space. Therefore, the whole map f has a lifting $\tilde{f}$. The ambiguity of the lifting is up to the μ_r action. q.e.d.

6.4. **Orbifold Gromov-Witten invariants on W^s.** We study the Gromov-Witten invariants that are needed in this paper.

Given a moduli space $\overline{\mathcal{M}}_{0,l,k}(W^s, d[\Gamma^s], \mathbf{x})$, one can define the Gromov-Witten invariants via evaluation maps:

$$ev_i : \overline{\mathcal{M}}_{0,l,k}(W^s, d[\Gamma^s], \mathbf{x}) \to X, 1 \le i \le l;$$
$$ev_j^{orb} : \overline{\mathcal{M}}_{0,l,k}(W^s, d[\Gamma^s], \mathbf{x}) \to Y_j, 1 \le j \le k.$$

The Gromov-Witten invariants are given by

$$\begin{aligned}&\Psi^{W^s}_{(d[\Gamma^s],0,l,k,\mathbf{x})}(\alpha_1, \dots, \alpha_l, \gamma_1, \dots, \gamma_k)\\ &= \int_{[\overline{\mathcal{M}}_{0,l,k}(W^s,d[\Gamma^s],\mathbf{x})]^{vir}} \bigcup_i ev_i^*(\alpha_i) \cup \bigcup_j ev_j^{orb,*}(\beta_j).\end{aligned}$$

Here $\alpha_i \in H^*(X)$ and $\beta_j \in H^*(Y_j)$. Note that l, k and $\mathbf{x}$ are specified by the α_i and β_j. For the sake of simplicity and consistency, we also re-denote the invariants by

$$\Psi^{W^s}_{(d[\Gamma^s],0,l+k)}(\alpha_1, \dots, \alpha_l, \gamma_1, \dots, \gamma_k),$$

when the α_i and β_j are given.

Lemma 6.7. *For $k \ge 1$ and $d \ge 1$*

$$\Psi^{W^s}_{(d[\Gamma^s],0,0,k,\mathbf{x})} = 0.$$

Proof. As explained in Lemma 6.2, this moduli space has negative dimension. Therefore the Gromov-Witten invariants have to be 0. q.e.d.

Proposition 6.8. *For $d \ge 1$, if $r \nmid d$, $\Psi^{W^s}_{(d[\Gamma^s],0)}$ vanishes. Otherwise, if $d = mr$*

$$\Psi^{W^s}_{(mr[\Gamma^s],0)} = \frac{1}{m^3}.$$

Proof. We have shown that

$$\overline{\mathcal{M}}_0(W^s, mr[\Gamma^s]) = \overline{\mathcal{M}}_0(\tilde{W}^s, m[\tilde{\Gamma}^s])/\mu_r.$$

This would suggest that

$$\Psi^{W^s}_{(mr[\Gamma^s],0)} = \frac{1}{r}\Psi^{\tilde{W}^s}_{(m[\tilde{\Gamma}^s],0)}. \tag{6.1}$$

This has to be shown by virtual techniques. Following the standard construction of virtual neighborhoods of moduli spaces, we have a smooth virtual moduli space

$$\mathcal{U}_0(\tilde{W}^s, m[\tilde{\Gamma}^s]) \supset \overline{\mathcal{M}}_0(\tilde{W}^s, m[\tilde{\Gamma}^s]),$$

with an obstruction bundle $\tilde{\mathcal{O}}$. The Gromov-Witten invariant is then given by

$$\Psi^{\tilde{W}^s}_{(m[\tilde{\Gamma}^s],0)} = \int_{\mathcal{U}_0(\tilde{W}^s, m[\tilde{\Gamma}^s])} \Theta(\tilde{\mathcal{O}}).$$

Here $\Theta(\tilde{\mathcal{O}})$ is the Thom form of the bundle. See the construction of virtual neighborhood in [CL] (and orginally in [R2]). The construction of virtual neighborhoods for $\overline{\mathcal{M}}_0(W^s, mr[\Gamma^s])$ is parallel. We also have

$$\mathcal{U}_0(W^s, mr[\Gamma^s])$$

with obstruction bundle $\mathcal{O}$. The model can be suggestively expressed as

$$(\mathcal{U}_0(W^s, mr[\Gamma^s]), \mathcal{O}) = (\mathcal{U}_0(\tilde{W}^s, m[\tilde{\Gamma}^s]), \tilde{\mathcal{O}})/\mu_r.$$

Therefore, we conclude that

$$\Psi^{W^s}_{(mr[\Gamma^s],0)} = \frac{1}{r}\int_{\mathcal{U}_0(\tilde{W}^s, m[\tilde{\Gamma}^s])} \Theta(\tilde{\mathcal{O}}) = \frac{1}{r}\Psi^{\tilde{W}^s}_{(m[\tilde{\Gamma}^s],0)}.$$

On the other hand,

$$\Psi^{\tilde{W}^s}_{(m[\tilde{\Gamma}^s],0)} = \frac{r}{m^3}.$$

This is computed in [BKL]. Therefore the proposition is proved. q.e.d.

6.5. **3-point functions on $H^*_{CR}(W^s)$ and $H^*_{CR}(W^{sf})$.** On W^s,

$$H^*_{CR}(W^s) = \mathbb{C}[1] \oplus \mathbb{C}(H^s) \oplus \bigoplus_{i=1}^{r-1} \mathbb{C}[\mathfrak{p}^s]_i \oplus \bigoplus_{j=1}^{r-1} \mathbb{C}[\mathfrak{q}^s]_j.$$

Given $\beta_i, 1 \leq i \leq 3$, in $H^*_{CR}(X)$ one defines the 3-point function as following:

$$\Psi^{W^s}(\beta_1, \beta_2, \beta_3) = \Psi^{W^s}_{CR}(\beta_1, \beta_2, \beta_3) + \sum_{d \geq 1} \Psi^{W^s}_{(d[\Gamma^s],0,3)}(\beta_1, \beta_2, \beta_3) q^{d[\Gamma^s]}.$$

Here the first term

$$\Psi^{W^s}_{CR}(\beta_1, \beta_2, \beta_3) = \Psi^{W^s}_{([0],0,3)}(\beta_1, \beta_2, \beta_3)$$

is the 3-point function defining the Chen-Ruan product. In the smooth case, this is just

$$\int \beta_1 \wedge \beta_2 \wedge \beta_3.$$

A similar expression for the orbifold case still holds. This is proved in [CH]: by introducing twisting factors, one can turn a twisted form β on twisted sector into a formal form $\tilde\beta$ on the global orbifold. Then we still have

$$\Psi^{W^s}_{CR}(\beta_1,\beta_2,\beta_3)=\int^{orb}_{W^s}\tilde\beta_1\wedge\tilde\beta_2\wedge\tilde\beta_3.$$

Remark 6.9. *Unfortunately, for the local model, $\Psi^{W^s}_{cr}(\beta_1,\beta_2,\beta_3)$ does not make sense if and only if all β_i are smooth classes, for the moduli space of the latter case is non-compact. Hence $\Psi^{W_s}_{CR}(\beta_1,\beta_2,\beta_3)$ is only a notation at the moment. But we will need it when we move on to study compact symplectic conifolds.*

By the computation in §6.4, we have

Proposition 6.10. *If at least one of the β_i is a twisted class,*

$$\Psi^{W^s}(\beta_1,\beta_2,\beta_3)=\Psi^{W_s}_{CR}(\beta_1,\beta_2,\beta_3).$$

Proof. Case 1, if all β_i are twisted classes,

$$\Psi^{W_s}_{(d[\Gamma^s],0,3)}(\beta_1,\beta_2,\beta_3)=0$$

if $d\geq 1$.

Now suppose β_3 is not twisted and the other two are.

Case 2: Suppose $\beta_3=1$; then it is well known that

$$\Psi^{W_s}_{(d[\Gamma^s],0,3)}(\beta_1,\beta_2,1)=0$$

if $d\geq 1$.

Case 3: suppose that $\beta_3=nH^s$; then

$$\Psi^{W_s}_{(d[\Gamma^s],0,3)}(\beta_1,\beta_2,\beta_3)=\beta_3(d[\Gamma^s])\Psi^{W_s}_{(d[\Gamma^s],0,2)}(\beta_1,\beta_2)=0.$$

Similar arguments can be applied to the case in which only one of the β_i is twisted. Hence the claim follows. q.e.d.

Now suppose $\deg(\beta_i)=2$, i.e. $\beta_i=n_iH^s$. Then

$$\sum_{m\geq 1}\Psi^{W_s}_{(mr[\Gamma^s],0,3)}(\beta_1,\beta_2,\beta_3)q^{mr[\Gamma_s]}=\beta_1([r\Gamma^s])\beta_2([r\Gamma^s])\beta_3([r\Gamma^s])\frac{q^{[r\Gamma^s]}}{1-q^{[r\Gamma^s]}}.$$

The last statement follows from Proposition 6.8. Hence

$$\Psi^{W^s}(\beta_1,\beta_2,\beta_3)=\int^{orb}_{W^s}\beta_1\wedge\beta_2\wedge\beta_3+\beta_1([r\Gamma^s])\beta_2([r\Gamma^s])\beta_3([r\Gamma^s])\frac{q^{[r\Gamma^s]}}{1-q^{[r\Gamma^s]}}.$$

Formally, we write $[\tilde\Gamma^s]=[r\Gamma_s]$. To summarize,

Proposition 6.11. *The three-point function* $\Psi^{W^s}(\beta_1, \beta_2, \beta_3)$ *of* W^s *is*

$$\Psi^{W^s}_{CR}(\beta_1, \beta_2, \beta_3)$$

if at least one of the β_i *is twisted or of degree 0, or*

$$\Psi^{W^s}_{cr}(\beta_1, \beta_2, \beta_3) + \beta_1(\tilde{\Gamma}^s)\beta_2(\tilde{\Gamma}^s)\beta_3(\tilde{\Gamma}^s)\frac{q^{[\tilde{\Gamma}^s]}}{1 - q^{[\tilde{\Gamma}^s]}},$$

if $\deg(\beta_i) = 2, 1 \leq i \leq 3$.

This proposition says that the quantum product $\beta_1 \star \beta_2$ is the usual product(in the sense of the Chen-Ruan ring structure) except for the case in which $\deg(\beta_1) = \deg(\beta_2) = 2$. Next, we write down the Chen-Ruan ring structure for twisted classes:

Proposition 6.12. *The Chen-Ruan products for twisted classes are given by*

$$[\mathfrak{p}^s]_i \star [\mathfrak{q}^s]_j = 0,$$
$$[\mathfrak{p}^s]_i \star [\mathfrak{p}^s]_j = \delta_{i+j,r}\Theta_{\mathbf{p}},$$
$$[\mathfrak{q}^s]_i \star [\mathfrak{q}^s]_j = \delta_{i+j,r}\Theta_{\mathbf{q}}.$$

Here Θ_p *and* Θ_q *are Thom forms of the normal bundles of* $\mathbf{p}$ *and* $\mathbf{q}$ *in* W^s*. Also*

$$\beta \star H^s = 0$$

if β *is a twisted class.*

Proof. This follows from the theorem in [CH]. As an example, we verify

$$[\mathfrak{p}^s]_i \star [\mathfrak{p}^s]_j = \delta_{i+j,r}\Theta_{\mathbf{p}} = 0.$$

For other cases, the proof is similar. The normal bundle of $\mathbf{p}$ is a rank 3 orbi-bundle which splits as three lines $\mathbb{C}_p, \mathbb{C}_y$ and $\mathbb{C}_z$ (cf. S2.3). Let Θ_p, Θ_y and Θ_z be the corresponding Thom forms. Then the twisting factor(cf. [CH]) of $[\mathfrak{p}^s]_i$ is

$$\mathfrak{t}([\mathfrak{p}^s]_i) = \Theta_p^b \Theta_y^{r-b} \Theta_z^i.$$

Here $b \equiv ai (\mod r)$ is an integer between 0 and $r-1$. Similarly, we write

$$\mathfrak{t}([\mathfrak{p}^s]_j) = \Theta_p^c \Theta_y^{r-c} \Theta_z^j.$$

Here $c \equiv aj (\mod r)$ is an integer between 0 and $r-1$. Then we have a formal computation

$$[\mathfrak{p}^s]_i \star [\mathfrak{p}^s]_j = \mathfrak{t}([\mathfrak{p}^s]_i) \wedge \mathfrak{t}([\mathfrak{p}^s]_j) = \delta_{i+j,r}\Theta_{\mathbf{p}}.$$

q.e.d.

Equivalently, this can be restated in terms of $\Psi^{W^s}_{cr}$ as

Proposition 6.13. *Suppose at least one of the β_i is twisted in the three-point function $\Psi^{W^s}_{cr}(\beta_1,\beta_2,\beta_3)$. Then only the following functions are nontrivial:*

$$\begin{aligned}\Psi^{W^s}_{cr}([\mathfrak{p}^s]_i,[\mathfrak{p}^s]_j,1) &= \delta_{i+j,r}\frac{1}{r};\\ \Psi^{W^s}_{cr}([\mathfrak{q}^s]_i,[\mathfrak{q}^s]_j,1) &= \delta_{i+j,r}\frac{1}{r}.\end{aligned}$$

6.6. **Identification of three-point functions Ψ^{W^s} and $\Psi^{W^{sf}}$.** We follow the argument in [LR]. Define a map

$$\phi: H^*_{CR}(W^{sf})\to H^*_{CR}(W^s).$$

On twisted classes, we define

$$\phi([\mathfrak{p}^{sf}]_k)=[\mathfrak{p}^s]_k,\ \ \phi([\mathfrak{q}^{sf}]_k)=[\mathfrak{q}^s]_k.$$

And on $H^*_{CR}(W^{sf})$, ϕ is defined as in the smooth case in [LR]. Since at the moment we are working in the local model, we should avoid using Poincare duality. We give a direct geometric construction of the map. On the other hand, a technical issue mentioned in Remark 6.9 is dealt with: let $\beta^{sf}_i, 1\le i\le 3$, be 2-forms on W^{sf} representing the classes $[\beta^{sf}_i]$; by the identification of $W^{sf}-\Gamma^{sf}$ with $W^s-\Gamma^s$, we then also have 2-forms in $W^s-\Gamma^s$ which as cohomology classes can be *uniquely* extended over W^s. The cohomology classes are denoted by

$$[\alpha_i]=\phi([\beta_i]).$$

Moreover we can require that the representing forms, denoted by α_i, coincide with β_i away from the Γ's.

Then we can define

$$\begin{aligned}\Psi^{W^s}_{CR}([\alpha_1],[\alpha_2],[\alpha_3]) &- \Psi^{W^{sf}}_{CR}([\beta_1],[\beta_2],[\beta_3])\\ &:= \int^{orb}_{W^s}\alpha_1\wedge\alpha_2\wedge\alpha_3-\int^{orb}_{W^{sf}}\beta_1\wedge\beta_2\wedge\beta_3.\end{aligned}$$

The well-definedness can be easily seen due to the coincidence of the α_i and β_i outside a compact set. Moreover,

Lemma 6.14. *Suppose that* $\deg\beta_i=2$*; then*

$$\begin{aligned}\Psi^{W^s}_{CR}([\alpha_1],[\alpha_2],[\alpha_3])-\Psi^{W^{sf}}_{CR}([\beta_1],[\beta_2],[\beta_3]) &= \alpha_1(\tilde\Gamma^s)\alpha_2(\tilde\Gamma^s)\alpha_3(\tilde\Gamma^s)\\ &= -\beta_1(\tilde\Gamma^{sf})\beta_2(\tilde\Gamma^{sf})\beta_3(\tilde\Gamma^{sf}).\end{aligned}$$

Proof. We lift the problem to $\tilde W^s$ and $\tilde W^{sf}$. Then we can further deform both models simultaneously to $\tilde V^s$ and $\tilde V^{sf}$ as [F]. Each of them consists r copies of the standard model $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$. $\tilde V^{sf}$ is a flop of $\tilde V^s$. Therefore, the computations are essentially r copies of

the computation on the standard model. By the argument in [LR], we have

$$\begin{aligned}\int^{orb}_{W^s}\alpha_1\wedge\alpha_2\wedge\alpha_3 &- \int^{orb}_{W^{sf}}\beta_1\wedge\beta_2\wedge\beta_3\\ &= \frac{1}{r}\left(\int_{\tilde W^s}\alpha_1\wedge\alpha_2\wedge\alpha_3-\int_{\tilde W^{sf}}\beta_1\wedge\beta_2\wedge\beta_3\right)\\ &= \frac{1}{r}\cdot r\cdot\alpha_1(\tilde\Gamma^s)\alpha_2(\tilde\Gamma^s)\alpha_3(\tilde\Gamma^s)\\ &= \alpha_1(\tilde\Gamma^s)\alpha_2(\tilde\Gamma^s)\alpha_3(\tilde\Gamma^s).\end{aligned}$$

Now we conclude that

Theorem 6.15. *Let $\beta_i\in H^*_{CR}(W^{sf}), 1\le i\le 3,$ and $\alpha_i=\phi(\beta_i)$. Then*

$$\Psi^{W^s}(\alpha_1,\alpha_2,\alpha_3)=\Psi^{W^{sf}}(\beta_1,\beta_2,\beta_3)$$

with the identification of $[\Gamma^s]\leftrightarrow-[\Gamma^{sf}]$.

Proof. The only nontrivial verification is for all $\deg\beta_i=2$. Suppose this is the case. Then the difference

$$\Psi^{W^s}(\alpha_1,\alpha_2,\alpha_3)-\Psi^{W^{sf}}(\beta_1,\beta_2,\beta_3)$$

includes two parts. Part(I) is

$$\Psi^{W^s}_{cr}([\alpha_1],[\alpha_2],[\alpha_3])-\Psi^{W^{sf}}_{cr}([\beta_1],[\beta_2],[\beta_3])=\alpha_1(\tilde\Gamma^s)\alpha_2(\tilde\Gamma^s)\alpha_3(\tilde\Gamma^s)$$

and part(II) is

$$\begin{aligned}&\alpha_1(\tilde\Gamma^s)\alpha_2(\tilde\Gamma^s)\alpha_3(\tilde\Gamma^s)\frac{q^{[\tilde\Gamma^s]}}{1-q^{[\tilde\Gamma^s]}}-\beta_1(\tilde\Gamma^{sf})\beta_2(\tilde\Gamma^{sf})\beta_3(\tilde\Gamma^{sf})\frac{q^{[\tilde\Gamma^{sf}]}}{1-q^{[\tilde\Gamma^{sf}]}}\\ =\;&\alpha_1(\tilde\Gamma^s)\alpha_2(\tilde\Gamma^s)\alpha_3(\tilde\Gamma^s)\frac{q^{[\tilde\Gamma^s]}}{1-q^{[\tilde\Gamma^s]}}+\alpha_1(\tilde\Gamma^s)\alpha_2(\tilde\Gamma^s)\alpha_3(\tilde\Gamma^s)\frac{q^{[-\tilde\Gamma^s]}}{1-q^{[-\tilde\Gamma^s]}}\\ =\;&-\alpha_1(\tilde\Gamma^s)\alpha_2(\tilde\Gamma^s)\alpha_3(\tilde\Gamma^s).\end{aligned}$$

Here we use $[\Gamma^s]\leftrightarrow-[\Gamma^{sf}]$. Part(I) cancels part (II), therefore

$$\Psi^{W^s}(\alpha_1,\alpha_2,\alpha_3)=\Psi^{W^{sf}}(\beta_1,\beta_2,\beta_3).$$

q.e.d.

7. Ruan's conjecture on orbifold symplectic flops

7.1. **Ruan cohomology.** Let X and Y be compact symplectic orbifolds related by symplectic flops. Correspondingly, Γ^s_i and Γ^{sf}_i, $1\le$

$i \leq k$, are extremal rays on X and Y respectively. We define three-point functions on X (similarly on Y):

$$\Psi^X_{qc}(\beta_1, \beta_2, \beta_3) = \Psi^X_{CR}(\beta_1, \beta_2, \beta_3) + \sum_{i=1}^{k} \sum_{d=1}^{\infty} \Psi^X_{(d[\Gamma^s_i],0,3)}(\beta_1, \beta_2, \beta_3).$$

This induces a ring structure on $H^*_{CR}(X)$

Definition 7.1. *Define the product on $H^*_{CR}(X)$ by*

$$\langle \beta_1 \star_r \beta_2, \beta_3 \rangle = \Psi^X_{qc}(\beta_1, \beta_2, \beta_3).$$

We call this the Ruan product on X. This cohomology ring is denoted by $RH_{CR}(X)$.

Similarly, we can define $RH^*_{CR}(Y)$ by using the three-point functions given by Γ^{sf}_i. Ruan conjectures that

Conjecture 7.1 (Ruan)**.** *$RH^*_{CR}(X)$ is isomorphic to $RH^*_{CR}(Y)$.*

7.2. **Verification of Ruan's conjecture.** Set

$$\Phi([\Gamma^s_u]) = -[\Gamma^{sf}_u].$$

This induces an obvious identification

$$\Phi : H_2(X) \to H_2(Y).$$

As explained in the local model, there is a natural isomorphism

$$\phi : H^*_{CR}(Y) \to H^*_{CR}(X).$$

We explain ϕ. For twisted classes $[\mathfrak{p}^{sf}_s]_i$ and $[\mathfrak{q}^{sf}_t]_j$ we define

$$\phi([\mathfrak{p}^{sf}_u]_i) = [\mathfrak{p}^s_u]_i, \quad , \phi([\mathfrak{q}^{sf}_v]_j) = [\mathfrak{q}^s_v]_j.$$

For degree 0 or 6-forms, ϕ is defined in an obvious way. For $\alpha \in H^2_{orb}(Y)$, $\phi(\alpha)$ is defined to be the unique extension of

$$\alpha|_{X - \cup \Gamma^s_u} = \alpha|_{Y - \cup \Gamma^{sf}_v}$$

over X. For $\beta \in H^4(Y)$, define $\phi(\beta) \in H^4(X)$ to be the extension as above such that

$$\int_X \phi(\beta) \wedge \phi(\alpha) = \int_Y \beta \wedge \alpha,$$

for any $\alpha \in H^2(Y)$. Then

Theorem 7.2. *For any classes $\beta_i \in H^*_{CR}(Y), 1 \leq i \leq 3$,*

$$\Phi_*(\Psi^X_{qc,r}(\phi(\beta_1), \phi(\beta_2), \phi(\beta_3))) = \Psi^Y_{qc,r}(\alpha_1, \alpha_2, \alpha_3).$$

Proof. If one of β_i, say β_1, has degree ≥ 4, the quantum correction term vanishes. Therefore, we need only verify

$$\Psi^X_{CR}(\phi(\beta_1), \phi(\beta_2), \phi(\beta_3)) = \Psi^Y_{CR}(\alpha_1, \alpha_2, \alpha_3).$$

We choose β_1 to be supported away from the Γ^{sf}. Then we have following observations:

- whenever β_2 or β_3 is a twisted class, both sides are equal to 0;
- if β_2 and β_3 are in $H^*(Y)$, then

$$\begin{aligned} \Psi^X_{cr}(\phi(\beta_1), \phi(\beta_2), \phi(\beta_3)) &= \int_X \phi(\beta_1) \wedge \phi(\beta_2) \wedge \phi(\beta_3) \\ &= \int_Y \beta_1 \wedge \beta_2 \wedge \beta_3 = \Psi^Y_{cr}(\alpha_1, \alpha_2, \alpha_3). \end{aligned}$$

Now we assume that β_i are either twisted classes or degree 2 classes. Then the verification is exactly same as that in Theorem 6.15. q.e.d.

As an corollary, we have proved

Theorem 7.3. *Suppose X and Y are related via an orbifold symplectic flops, Via the map ϕ and coordinate change Φ,*

$$RH^*_{CR}(X) \cong RH^*_{CR}(Y).$$

This explicitly realizes the claim of Theorem 1.3.

References

[BKL] J. Bryant, S. Katz, N. Leung, Multiple covers and the integrality conjecture for rational curves in CY threefolds , J. ALgebraic Geometry 10(2001),no.3.,549-568.

[CH] B. Chen, S. Hu, A de Rham model of Chen-Ruan cohomology ring of abelian orbifolds, To appear in Math. Ann.

[CL] B. Chen, A-M. Li, Symplectic Virtual Localization of Gromov-Witten invariants, in preparation.

[CLZZ] B. Chen, A-M. Li, Q. Zhang, G. Zhao, Relative Gromov-Witten invariants and glue formula, in preparation.

[1] CT B. Chen, G. Tian, Virtual orbifolds and Localization, preprint.

[CR1] W. Chen, Y. Ruan, A new cohomology theory for orbifold, AG/0004129, Commun. Math. Phys., 248(2004), 1-31.

[CR2] W. Chen, Y. Ruan, orbifold Gromov-Witten theory, AG/0011149. Cont. Math., 310, 25-86.

[CR3] W. Chen, Y. Ruan, orbifold quantum cohomology, Preprint AG/0005198.

[F] R. Friedman, Simultaneous resolutions of threefold double points, Math. Ann. 274(1986) 671-689.

[Gr] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, Invent. math., 82 (1985), 307-347.

[HZ] J. Hu, W. Zhang, Mukai flop and Ruan cohomology, Math. Ann. 330, No.3, 577-599 (2004).

[K] J. Kollár, Flips, Flops, Minimal Models, Etc., Surveys in Differential Geometry, 1(1991),113-199.

[La] Henry B. Laufer, On CP^1 as an xceptional set, In recent developments in several complex variables ,261-275, Ann. of Math. Studies 100, Princeton, 1981.

[L] E. Lerman, Symplectic cuts, Math Research Let 2(1985) 247-258

[LR] A-M. Li, Y. Ruan, Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds, Invent. Math. 145, 151-218(2001)

[LZZ] A-M. Li, G. Zhao, Q. Zheng, The number of ramified covering of a Riemann surface by Riemann surface, Commu. Math. Phys, 213(2000), 3, 685–696.

[Reid] M. Reid, Young Person's Guide to Canonical Singularities, Proceedings of Symposia in Pure Mathematics, V.46 (1987).

[R] Y. Ruan, Surgery, quantum cohomology and birational geometry, math.AG/9810039.

[R2] Y. Ruan, Virtual neighborhoods and pseudo-holomorphic curves, alg-geom/9611021 ,

[S] I. Satake, The Gauss-Bonnet theorem for V-manifolds, J. Math. Soc. Japan 9(1957), 464-492.

[STY] I. Smith, R.P. Thomas, S.-T. Yau, Symplectic conifold transitions. SG/0209319. J. Diff. Geom., 62(2002), 209-232.

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